

Covering a cubic graph by 5 perfect matchings

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Abstract

Berge Conjecture states that every bridgeless cubic graph has 5 perfect matchings such that each edge is contained in at least one of them. In this paper, we show that Berge Conjecture holds for two classes of cubic graphs, cubic graphs with a circuit missing only one vertex and bridgeless cubic graphs with a 2-factor consisting of two circuits. The first part of this result implies that Berge Conjecture holds for hypohamiltonian cubic graphs.

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1 Introduction

Graphs in this article may contain multiple edges but contain no loops. A k -factor of a graph G is a spanning k -regular subgraph of G . The set of edges in a 1-factor of a graph G is called a *perfect matching* of G . A *matching* of a graph G is a set of edges in a 1-regular subgraph of G . A *perfect matching cover* of a graph G is a set of perfect matchings of G such that each edge of G is contained in at least one member of it. The *order* of a perfect matching cover is the number of perfect matchings in it.

One of the first theorems in graph theory, Petersen's Theorem from 1891 [18], states that every bridgeless cubic graph has a perfect matching. By Tutte's Theorem from 1947 [20], which states that a graph G has a perfect matching if and only if the number of odd components of $G - X$ is not greater than the size of X for all $X \subseteq V(G)$, we can obtain

that every edge in a bridgeless cubic graph G is contained in a perfect matching of G . This implies that every bridgeless cubic graph has a perfect matching cover. What is the minimum number k such that every bridgeless cubic graph has a perfect matching cover of order k ? Berge conjectured this number is 5 (unpublished, see e.g. [7, 8]).

Conjecture 1.1 (Berge Conjecture). *Every bridgeless cubic graph has a perfect matching cover of order at most 5.*

The following conjecture is attributed to Berge in [19], and was first published in an paper by Fulkerson [4].

Conjecture 1.2 (Fulkerson Conjecture). *Every bridgeless cubic graph has six perfect matchings such that each edge belongs to exactly two of them.*

Mazzuoccolo [8] proved that Conjectures 1.1 and 1.2 are equivalent. The equivalence of these two conjectures does not imply that Conjecture 1.2 holds for a given bridgeless cubic graph satisfying Conjecture 1.1. It is still open question whether this holds.

A cubic graph G is called *3-edge-colorable* if G has three edge-disjoint perfect matchings. It is trivial that Conjectures 1.1 and 1.2 hold for 3-edge-colorable cubic graphs. Non-3-edge-colorable and cyclically 4-edge-connected cubic graphs with girth at least 5 are called *snarks*. Conjecture 1.2 have been verified for some families of snarks, such as flower snarks, Goldberg snarks, generalised Blanuša snarks, and Loupekinine snarks [6, 13, 15].

Besides the above snarks, some families of cubic graphs have been confirmed to satisfy Conjecture 1.1. Steffen [17] showed that Conjecture 1.1 holds for bridgeless cubic graphs which have no nontrivial 3-edge-cuts and have 3 perfect matchings which miss at most 4 edges. It is proved by Hou et al. [11] that every almost Kotzig graph has a perfect matching cover of order 5. Esperet and Mazzuoccolo [3] showed that there are infinite cubic graphs of which every perfect matching cover has order at least 5 and the problem that deciding whether a bridgeless cubic graph has a perfect matching cover of order at most 4 is NP-complete.

In this paper, we show that Berge Conjecture holds for a cubic graph which has a vertex whose removal results a hamiltonian graph. This implies that Berge Conjecture holds for hypohamiltonian cubic graphs, a class of cubic graphs which was conjectured to satisfy Fulkerson Conjecture by Höggkvist [9]. A graph G is called *hypohamiltonian* if G itself is not hamiltonian but the removal of any vertex of G results a hamiltonian graph. Chen and Fan [1] verified the Fulkerson Conjecture for several known classes of hypohamiltonian graphs in the literatures. Now Höggkvist's conjecture is still open.

In this paper, we also show that Berge Conjecture holds for bridgeless cubic graphs with a 2-factor consisting of two circuits. This class of cubic graphs include permutation graphs and permutation graphs include generalized Petersen graphs. Fouquet and Vanherpe [7] showed that every permutation graph have a perfect matching cover of order 4. It was proved by Castagna, Prins [2] and Watkins [21] that all generalized Petersen graphs but the original Petersen graph are 3-edge-colorable.

2 A technical lemma

Some notations will be used in this paper. Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. For $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X and denote by $G - X$ the subgraph of G induced by $V(G) \setminus X$. For $F \subseteq E(G)$, we denote by $G[F]$ the subgraph induced by F and denote by $G - F$ the subgraph of G with vertex-set $V(G)$ and edge-set $E(G) \setminus F$. For $F_1, F_2 \subseteq E(G)$, we denote by $F_1 \triangle F_2$ the set $(F_1 \setminus F_2) \cup (F_2 \setminus F_1)$. A path P of length at least 1 in G is called a F_1 - F_2 *alternating* path of G if $E(P) \subseteq F_1 \cup F_2$ and each of $E(P) \cap F_1$ and $E(P) \cap F_2$ is a matching of G . For a positive integer n , we denote by $[n]$ the set $\{1, 2, \dots, n\}$.

Now we present a technical lemma, which plays a key role in the proof of our main results.

Lemma 2.1. *Let G be a cubic graph which has three edge-disjoint perfect matchings M_1 , M_2 and M_3 such that both $M_1 \cup M_2$ and $M_1 \cup M_3$ induce hamiltonian circuits of G . Let F be a non-empty subset of M_2 and α be an edge in M_3 . We have that G has two perfect matchings M_4 and M_5 such that*

- (1) *either $M_4 \cap M_5 \subseteq M_3 \subseteq M_4 \cup M_5$ or $M_4 \cap M_5 \subseteq M_1 \subseteq M_4 \cup M_5$,*
- (2) *$G[M_4 \cup M_5]$ has a circuit C containing α such that $F \cap E(C) \neq \emptyset$ and every circuit different from C in $G[M_4 \cup M_5]$ contains no edges in F , and*
- (3) *if $M_1 \subseteq M_4 \cup M_5$, then G has a circuit C' containing α such that $M_2 \cap E(C) \cap E(C') = \emptyset$, $M_3 \setminus (M_4 \cup M_5) \subseteq M_3 \setminus E(C')$ and $M_3 \setminus E(C')$ is a perfect matching of $G - V(C')$.*

Proof. We proceed by induction on $|V(G)|$. If $|V(G)| = 2$, then M_2 and M_3 meet the requirements. So the statement holds for $|V(G)| = 2$. Now we suppose $|V(G)| > 2$.

Let C_1 be the circuit containing α in $G[M_2 \cup M_3]$. If $F \subseteq E(C_1)$, then M_2 and M_3 meet the requirements. So we assume $F \setminus E(C_1) \neq \emptyset$.

Set $E_1 := (M_3 \setminus E(C_1)) \cup (M_2 \cap E(C_1))$. We know that every component of $G - E_1$ is a even circuit. Let C_2 be the circuit containing α in $G - E_1$. Let M_4 and M_5 be the two

edge-disjoint perfect matchings of $G - E_1$. We know $M_4 \cap M_5 = \emptyset \subseteq M_1 \subseteq M_4 \cup M_5$, $M_2 \cap E(C_2) \cap E(C_1) = \emptyset$, $M_3 \setminus (M_4 \cup M_5) \subseteq M_3 \setminus E(C_1)$ and that $M_3 \setminus E(C_1)$ is a perfect matching of $G - V(C_1)$. If $F \setminus E(C_1) \subseteq E(C_2)$, then M_4 and M_5 are two perfect matchings of G which meet the requirements. So we assume further $F \setminus (E(C_1) \cup E(C_2)) \neq \emptyset$.

Let C_3 be a circuit in $G - E_1$ such that $E(C_3) \cap (F \setminus (E(C_1) \cup E(C_2))) \neq \emptyset$. Set $E_2 := M_2 \setminus E(C_3)$. Let $P_{1,1}, P_{1,2}, \dots, P_{1,t}$ be the (inclusionwise) maximal M_1 - M_3 alternating paths in $G - E_2$ which contain no edges in C_3 . We know $|M_3 \cap E(P_{1,i})| = |M_1 \cap E(P_{1,i})| + 1$ for each $i \in [t]$. Since $G[M_1 \cup M_3]$ is a hamiltonian circuit of G , there is some $s \in [t]$ such that $\alpha \in E(P_{1,s})$. Let $P_{2,1}, P_{2,2}, \dots, P_{2,t}$ be the (inclusionwise) maximal M_1 - M_3 paths in C_3 . We know $|M_1 \cap E(P_{2,i})| = |M_3 \cap E(P_{2,i})| + 1$ for each $i \in [t]$. For each $i \in \{1, 2\}$ and each $j \in [t]$, let $\beta_{i,j}$ be an edge with the same ends as $P_{i,j}$. Set $M_6 := \{\beta_{2,j} : j \in [t]\}$, $M_7 := M_2 \cap E(C_3)$ and $M_8 := \{\beta_{1,j} : j \in [t]\}$. Set $F' := E(C_3) \cap (F \setminus (E(C_1) \cup E(C_2)))$.

We construct a new graph G' with vertex-set $V(G[M_7])$ and edge-set $M_6 \cup M_7 \cup M_8$. From above, we know that $M_6 \cup M_7$ induce a hamiltonian circuit of G' . Since $G[M_1 \cup M_3]$ is a hamiltonian circuit of G , $M_6 \cup M_8$ induce a hamiltonian circuit of G' . As $|V(G')| < |V(G)|$, we know by the induction hypothesis that G' has two perfect matchings M_9 and M_{10} such that (1) either $M_9 \cap M_{10} \subseteq M_8 \subseteq M_9 \cup M_{10}$ or $M_9 \cap M_{10} \subseteq M_6 \subseteq M_9 \cup M_{10}$, (2) $G'[M_9 \cup M_{10}]$ has a circuit C'_1 containing $\beta_{1,s}$ such that $F' \cap E(C'_1) \neq \emptyset$ and every circuit different from C'_1 in $G'[M_9 \cup M_{10}]$ contains no edges in F' , and (3) if $M_6 \subseteq M_9 \cup M_{10}$, then G' has a circuit C'_2 containing $\beta_{1,s}$ such that $M_7 \cap E(C'_1) \cap E(C'_2) = \emptyset$, $M_8 \setminus (M_9 \cup M_{10}) \subseteq M_8 \setminus E(C'_2)$ and $M_8 \setminus E(C'_2)$ is a perfect matching of $G' - V(C'_2)$.

Set $E_3 := \bigcup_{j=1}^2 (\bigcup_{k \in [t] \text{ s.t. } \beta_{j,k} \in M_9 \triangle M_{10}} E(P_{j,k}))$. Let M_{11} be M_3 if $M_8 \subseteq M_9 \cup M_{10}$ and be M_1 if $M_6 \subseteq M_9 \cup M_{10}$. Set $M_{12} := E_3 \triangle M_{11}$. Noting either $M_9 \cap M_{10} \subseteq M_8 \subseteq M_9 \cup M_{10}$ or $M_9 \cap M_{10} \subseteq M_6 \subseteq M_9 \cup M_{10}$, we have that M_{12} is a perfect matching of G and either $M_{11} \cap M_{12} \subseteq M_3 \subseteq M_{11} \cup M_{12}$ or $M_{11} \cap M_{12} \subseteq M_1 \subseteq M_{11} \cup M_{12}$. Let C_4 be the circuit of G which is obtained from C'_1 by replacing each edge $\beta_{j,k}$ in C'_1 by the corresponding path $P_{j,k}$. We can see from the property of C'_1 that C_4 is a circuit in $G[M_{11} \cup M_{12}]$ such that $\alpha \in E(C_4)$, $F \cap E(C_4) \neq \emptyset$ and every circuit different from C_4 in $G[M_{11} \cup M_{12}]$ contains no edges in F .

Suppose $M_1 \subseteq M_{11} \cup M_{12}$. We have $M_6 \subseteq M_9 \cup M_{10}$. Let C_5 be the circuit obtained from C'_2 by replacing each edge $\beta_{j,k}$ in C'_2 by the corresponding path $P_{j,k}$. As $\beta_{1,s} \in E(C'_2)$ and $M_7 \cap E(C'_1) \cap E(C'_2) = \emptyset$, we know $\alpha \in E(C_5)$ and $M_2 \cap E(C_4) \cap E(C_5) = \emptyset$. Since $M_8 \setminus E(C'_2)$ is a perfect matching of $G' - V(C'_2)$, $M_3 \setminus E(C_5)$ is a perfect matching of $G - V(C_5)$. Noting also $M_9 \cap M_{10} \subseteq M_6 \subseteq M_9 \cup M_{10}$ and $M_8 \setminus (M_9 \cup M_{10}) \subseteq M_8 \setminus E(C'_2)$, we have $M_3 \setminus (M_{11} \cup M_{12}) \subseteq M_3 \setminus E(C_5)$.

So M_{11}, M_{12} are perfect matchings of G which meet the requirements. \square

3 Main results

In this section, we show that Berge Conjecture holds for a bridgeless cubic graph which has a circuit missing only one vertex or has a 2-factor consisting of two circuit.

Lemma 3.1. *Let G a bridgeless cubic graph with a 2-factor consisting of two odd circuits C_1 and C_2 . Let u_1u_2 be an edge in G with $u_1 \in V(C_1)$ and $u_2 \in V(C_2)$ and let M be the perfect matching of G such that $u_1u_2 \in M$ and $M \setminus \{u_1u_2\} \subseteq E(C_1) \cup E(C_2)$. For $i = 1, 2$, let C_{i+2} be the circuit containing u_i in $G[E(G) \setminus M]$. Suppose $C_3 \neq C_4$ and that G has a circuit C containing u_1 such that*

- (1) $(E(C_1) \setminus M) \setminus E(C)$ is a perfect matching of $C_1 - (V(C) \cap V(C_1))$,
- (2) $\emptyset \neq E(C) \cap E(C_2) \subseteq E(C_2) \setminus M$ and $E(C) \cap E(C_2) \cap E(C_4) = \emptyset$, and
- (3) the paths Q_1, Q_2, \dots, Q_s separated by $E(C) \cap E(C_2)$ in C satisfy that for each $i \in [s]$, $E(Q_i) \cap (E(C_1) \setminus M)$ is a perfect matching of $Q_i - (V(Q_i) \cap V(C_2))$ if $u_1 \notin V(Q_i)$.

We have that G has 3 perfect matchings covering all edges in $(E(C_1) \cup E(C_2)) \setminus M$.

Proof. Set $M_1 := (E(C_1) \cup E(C_2)) \setminus M$, $M_2 := M$ and $M_3 := E(G) \setminus (E(C_1) \cup E(C_2))$. Let C_5 be the circuit containing u_1 in $G[E(C) \triangle E(C_2)]$. From the properties (1) and (3) of C , we know that $(E(C_1) \cap M_1) \setminus E(C_5)$ is a perfect matching of $C_1 - (V(C_5) \cap V(C_1))$.

Assume $u_2 \in V(C_5)$. Then every component of $G[E(C) \triangle E(C_2)]$ is an even circuit. So $E(C) \triangle E(C_2)$ can be decomposed into two matchings N_1 and N_2 of G . For $i = 4, 5$, set $M_i := N_{i-3} \cup ((E(C_1) \cap M_1) \setminus E(C))$. From the property (1) of C , we can know that M_4 and M_5 are perfect matchings of G and we can see $M_1 \setminus (M_4 \cup M_5) = E(C) \cap E(C_2)$. So it suffice to show that $E(C) \cap E(C_2)$ is contained in a perfect matching of G . On the other hand, it is easy to see that $G[\{u_1u_2\} \cup E(C_1) \cup E(C_4)]$ has a perfect matching, say N_3 . Noting $E(C) \cap E(C_2) \cap E(C_4) = \emptyset$, we have that $N_3 \cup ((E(C_2) \cap M_1) \setminus E(C_4))$ is a perfect matching of G which contains $E(C) \cap E(C_2)$.

Next we assume $u_2 \notin V(C_5)$. Let $P_{1,1}, P_{1,2}, \dots, P_{1,t}$ be the components of $G[E(C_5) \cap E(C_2)]$. We know that for each $i \in [t]$, $P_{1,i}$ is a M_1 - M_2 alternating path satisfying $|E(P_{1,i}) \cap M_2| = |E(P_{1,i}) \cap M_1| + 1$. For $i = 2, 3$, let $P_{i,1}, P_{i,2}, \dots, P_{i,t}$ be the paths in C_{11-3i} which are separated by $P_{1,1}, P_{1,2}, \dots, P_{1,t}$. We may assume $u_1 \in V(P_{2,1})$ and $u_2 \in V(P_{3,1})$. We know $P_{2,j} \in \{Q_1, Q_2, \dots, Q_s\}$ for each $j \in [t]$. For each $i \in \{1, 2, 3\}$ and each $j \in [t]$, let $\alpha_{i,j}$ be an edge with the same ends as $P_{i,j}$. Set $A_i := \{\alpha_{i,j} : j \in [t]\}$ for $i = 1, 2, 3$.

We construct a new graph G' whose vertex-set consists of the ends of edges in A_1 and edge-set is $A_1 \cup A_2 \cup A_3$. We know that both $A_1 \cup A_2$ and $A_1 \cup A_3$ induce hamiltonian circuits of G' . For $\alpha_{2,1} \in A_2$ and $\alpha_{3,1} \in A_3$, we have by Lemma 2.1 that G' has two perfect matchings F_1 and F_2 such that (1) either $F_1 \cap F_2 \subseteq A_3 \subseteq F_1 \cup F_2$ or $F_1 \cap F_2 \subseteq A_1 \subseteq F_1 \cup F_2$, (2) $G'[F_1 \cup F_2]$ has a circuit C'_1 containing $\alpha_{3,1}$ and $\alpha_{2,1}$, and (3) if $A_1 \subseteq F_1 \cup F_2$, then G' has a circuit C'_2 containing $\alpha_{3,1}$ such that $A_2 \cap E(C'_1) \cap E(C'_2) = \emptyset$, $A_3 \setminus (F_1 \cup F_2) \subseteq A_3 \setminus E(C'_2)$ and $A_3 \setminus E(C'_2)$ is a perfect matching of $G' - V(C'_2)$.

Set $E_1 := \bigcup_{i=1}^3 (\bigcup_{j \in [t] \text{ s.t. } \alpha_{i,j} \in F_1 \triangle F_2} E(P_{i,j}))$. From above, we can obtain that every component of $G[E_1]$ is an even circuit of G . Hence E_1 can be decomposed into two matchings N_4 and N_5 of G .

Assume $F_1 \cap F_2 \subseteq A_3 \subseteq F_1 \cup F_2$. We have that $A_3 \cap (F_1 \triangle F_2)$ is a perfect matching of $G'[F_1 \triangle F_2]$. It follows that $M_1 \setminus E_1$ is a perfect matching of $G - V(G[E_1])$. Hence $(M_1 \setminus E_1) \cup N_4$ and $(M_1 \setminus E_1) \cup N_5$ are two perfect matchings of G which cover all edges in M_1 .

Next we assume $F_1 \cap F_2 \subseteq A_1 \subseteq F_1 \cup F_2$. Set $E_2 := (\bigcup_{j \in [t] \text{ s.t. } \alpha_{1,j} \in F_1 \cap F_2} E(P_{1,j})) \cup (\bigcup_{j \in [t] \text{ s.t. } \alpha_{3,j} \in A_3 \setminus (F_1 \cup F_2)} E(P_{3,j}))$. We know $E_2 \subseteq M_1 \cup M_2$. For $i = 6, 7$, set $M_i := N_{i-2} \cup (E_2 \cap M_2) \cup ((M_1 \cap E(C_1)) \setminus E_1)$. We can see that M_6 and M_7 are perfect matchings of G and we have $M_1 \setminus (M_6 \cup M_7) = E_2 \cap M_1$.

Now we show that $E_2 \cap M_1$ is contained in a perfect matching of G . Let C_6 be the circuit of G which is obtained from C'_2 by replacing each edge $\alpha_{i,j}$ in C'_2 by the corresponding path $P_{i,j}$. Noting $\alpha_{2,1} \in E(C'_1)$, $\alpha_{3,1} \in E(C'_2)$ and $A_2 \cap E(C'_1) \cap E(C'_2) = \emptyset$, we have $u_2 \in V(C_6)$ and $u_1 \notin V(C_6)$. Notice that $A_3 \setminus (F_1 \cup F_2) \subseteq A_3 \setminus E(C'_2)$ and $A_3 \setminus E(C'_2)$ is a perfect matching of $G' - V(C'_2)$. It follows that $E_2 \cap M_1 \subseteq (M_1 \cap E(C_2)) \setminus E(C_6)$, $(M_1 \cap E(C_2)) \setminus E(C_6)$ is a perfect matching of $C_2 - (V(C_6) \cap V(C_2))$ and $E(C_6) \setminus M_1$ is the perfect matching of $C_6 - u_2$. If $E(C_6) \cap E(C_1) = \emptyset$, then $(M_2 \setminus E(C_2)) \cup ((M_1 \cap E(C_2)) \triangle E(C_6))$ is a perfect matching containing $E_2 \cap M_1$ in G . So we assume $E(C_6) \cap E(C_1) \neq \emptyset$. Then there is a path T from u_2 to $V(C_1)$ in C_6 such that $|V(T) \cap V(C_1)| = 1$. Let N_6 be the perfect matching of $C_1 - (V(T) \cap V(C_1))$. We know that $N_6 \cup ((M_1 \cap E(C_2)) \triangle E(T))$ is a perfect matching containing $E_2 \cap M_1$ in G .

So the edges in M_1 can be covered by 3 perfect matchings of G . □

Theorem 3.2. *Let G be a cubic graph. Suppose that G has a vertex v such that $G - v$ has a hamiltonian circuit. Then G has a perfect matching cover of order 5.*

Proof. Let C be a hamiltonian circuit in $G - v$. Choose a vertex u in $V(C)$ such that $uv \in E(G)$. Let N_1 be the perfect matching of $C - u$. Set $N_2 := E(C) \setminus N_1$. Let C_1 (C_2)

be the circuit containing u (v) in $G[E(G) \setminus (N_1 \cup \{uv\})]$. If $C_1 = C_2$, then every circuit in $G[E(G) \setminus (N_1 \cup \{uv\})]$ has even length, which implies that G is 3-edge-colorable and the statement holds. So we assume $C_1 \neq C_2$. Set $M_1 := N_1 \cup \{uv\}$ and $M_2 := (E(C_1) \setminus E(C)) \cup (E(C_2) \cap E(C)) \cup (E(G) \setminus (E(C) \cup E(C_1) \cup E(C_2)))$. We know that M_1 and M_2 be two perfect matchings of G .

Let P_1, P_2, \dots, P_t be the paths of length at least 1 in C , which are separated by $(E(C_1) \cup E(C_2)) \cap E(C)$. For each $i \in [t]$, we know that P_i is a N_1 - N_2 alternating path satisfying $|E(P_i) \cap N_1| = |E(P_i) \cap N_2| + 1$. For each $i \in [t]$, let α_i be an edge with the same ends as P_i . Let G' be a new graph with vertex-set $V(C_1) \cup V(C_2)$ and edge-set $\{uv\} \cup E(C_1) \cup E(C_2) \cup \{\alpha_i : i \in [t]\}$. We know that G' is a bridgeless cubic graph and $E(C_1) \cup E(C_2)$ induces a 2-factor of G' . Let M' be the perfect matching of G' such that $uv \in M'$ and $M' \setminus \{uv\} \subseteq E(C_1) \cup E(C_2)$. Let C_3 (C_4) be the circuit containing u (v) in $G'[E(G') \setminus M']$.

Assume $C_3 = C_4$. It implies that $G - M_2$ is a 2-factor of G which contains no odd circuits. Hence G is 3-edge-colorable and the statement holds.

Assume $C_3 \neq C_4$. Noting $E(C) \cap E(C_i) \neq \emptyset$ for $i = 1, 2$, we have that $G'[(E(C) \cap E(C_1)) \cup \{\alpha_i : i \in [t]\}]$ contains no circuits, which implies $E(C_3) \cap E(C_2) \neq \emptyset$. We can see easily that for the perfect matching M' of G' , C_3 is a circuit meeting the requirements (1)-(3) in Lemma 3.1. By Lemma 3.1, G' has 3 perfect matchings M'_3, M'_4 and M'_5 which cover all edges in $(E(C_1) \cup E(C_2)) \setminus M'$. We know $\{\alpha_i : i \in [t]\} \cap M'_3 \cap M'_4 \cap M'_5 = \emptyset$. For $i = 3, 4, 5$, set $M_i := (\bigcup_{j \in [t] \text{ s.t. } \alpha_j \in M'_i} (E(P_j) \cap N_1)) \cup (\bigcup_{j \in [t] \text{ s.t. } \alpha_j \notin M'_i} (E(P_j) \cap N_2)) \cup (M'_i \setminus \{\alpha_j : j \in [t]\})$. We have that M_1, M_2, M_3, M_4 and M_5 are 5 perfect matchings of G which cover all edges of G . \square

By Theorem 3.2, we can obtain immediately that Berge Conjecture holds for cubic hypohamiltonian graphs.

Corollary 3.3. *If G is a hypohamiltonian cubic graph, then G has a perfect matching cover of order 5.*

Theorem 3.4. *Let G be a bridgeless cubic graph with a 2-factor consisting of two circuits. Then G has a perfect matching cover of order 5.*

Proof. We know that G has two vertex-disjoint circuits C_1 and C_2 such that $V(G) = V(C_1) \cup V(C_2)$. If both C_1 and C_2 have even lengths, then G is 3-edge-colorable and the statement holds. So we assume that both C_1 and C_2 have odd lengths. Choose an edge $u_1 u_2 \in E(G)$ with $u_1 \in V(C_1)$ and $u_2 \in V(C_2)$. Set $M_3 := E(G) \setminus (E(C_1) \cup E(C_2))$ and

let M_2 be the perfect matching of G such that $M_2 \cap M_3 = \{u_1 u_2\}$. Set $M_1 := (E(C_1) \cup E(C_2)) \setminus M_2$. For $i = 1, 2$, let C_{i+2} be the circuit containing u_i in $G[E(G) \setminus M_2]$. If $C_3 = C_4$, then G is 3-edge-colorable and the statement holds. So we assume further $C_3 \neq C_4$.

Assume $E(C_3) \cap E(C_2) \neq \emptyset$. We can see that for the perfect matching M_2 of G , C_3 meets the requirements (1)-(3) in Lemma 3.1. By Lemma 3.1, the edges in M_1 can be covered by 3 perfect matchings of G , which together with M_2 and M_3 cover all edges of G .

So we assume $E(C_3) \cap E(C_2) = \emptyset$. Similarly, we can also assume $E(C_1) \cap E(C_4) = \emptyset$.

Since G is bridgeless, we know $|M_3| \geq 3$. It follows that there is a circuit C_5 in $G[E(G) \setminus M_2]$ such that $E(C_5) \cap E(C_1) \neq \emptyset$ and $E(C_5) \cap E(C_2) \neq \emptyset$. We know $V(C_5) \cap V(C_i) = \emptyset$ for $i = 3, 4$. Let Q be the (inclusionwise) maximal path containing u_1 in C_1 such that $E(Q) \cap E(C_5) = \emptyset$.

Claim 1. *G has a perfect matching containing $(M_1 \cap E(C_1)) \setminus (E(Q) \cup E(C_5))$.*

Set $E_1 := (M_1 \cap E(C_1)) \setminus (E(Q) \cup E(C_5))$. Let u_3 and u_4 be the ends of Q . For $i = 1, 2$, let β_i be the edge incident to u_{i+2} in C_5 . For $i = 1, 2$, let T_i be the path from u_{i+2} to $V(C_2) \cup \{u_{5-i}\}$ in C_5 such that $\beta_i \in E(T_i)$ and $|V(T_i) \cap (V(C_2) \cup \{u_{5-i}\})| = 1$. For $i = 1, 2$, let u_{i+4} be the end of T_i which is different from u_{i+2} .

Assume $u_5 \in V(C_2)$ or $u_6 \in V(C_2)$. Without loss of generality, we assume $u_5 \in V(C_2)$. Let T_3 be the path from u_1 to u_3 in Q . Let N_1 be the perfect matching of $C_2 - u_5$. Then $((M_1 \cap E(C_1)) \triangle (E(T_1) \cup E(T_3))) \cup N_1$ is a perfect matching containing E_1 in G .

Assume $u_5 = u_4$. If $\beta_2 \in E(T_1)$, then $(M_2 \setminus E(C_1)) \cup ((M_1 \cap E(C_1)) \triangle (E(Q) \cup E(T_1)))$ is a perfect matching containing E_1 in G . So we assume $\beta_2 \notin E(T_2)$. Noting $E(C_5) \cap E(C_2) \neq \emptyset$, we have $u_6 \in V(C_2)$. This returns to the case we have discussed in the previous paragraph. Claim 1 is proved.

In the following proof, if $P_{i,j}$ is a path of G , then let $\alpha_{i,j}$ be an edge with the same ends as $P_{i,j}$.

Claim 2. *If G has two circuits C and C' such that*

- (1) $u_1 \in V(C) \cap V(C')$,
- (2) $\emptyset \neq E(C) \cap E(C_2) \subseteq E(C_5) \cap E(C_2)$, $E(C') \cap E(C_2) \subseteq E(C_5) \cap E(C_2)$ and $E(C) \cap E(C') \cap E(C_2) = \emptyset$,
- (3) the paths Q_1, Q_2, \dots, Q_q separated by $E(C) \cap E(C_2)$ in C satisfy that $E(Q) \subseteq E(Q_1)$ and for each $i \in [q] \setminus \{1\}$, $M_2 \cap E(Q_i)$ is a perfect matching of $Q_i - (V(Q_i) \cap V(C_2))$,
- (4) $G[V(C_1)] - (V(C) \cap V(C_1))$ has two perfect matchings N_2 and N_3 satisfying $E(C_1) \setminus (E(C) \cup N_2 \cup N_3) \subseteq (M_1 \cap E(C_1)) \setminus E(C')$, and

(5) $(M_1 \cap E(C_1)) \setminus E(C')$ is a perfect matching of $C_1 - (V(C') \cap V(C_1))$ and $E(C') \setminus M_1$ is a perfect matching of $C' - u_1$,
then G has 5 perfect matchings which cover all edges of G .

Suppose G has such two circuits C and C' . Set $D_1 := E(C) \cap E(C_2)$. We know $D_1 \subseteq M_1$. Let $P_{1,1}, P_{1,2}, \dots, P_{1,q}$ be the paths in C_2 which are separated by D_1 . We may assume $u_2 \in V(P_{1,1})$. For each $i \in [q]$, let γ_i be an edge with the same ends as Q_i . Set $D_2 := \{\alpha_{1,j} : j \in [q]\}$ and $D_3 := \{\gamma_j : j \in [q]\}$. We construct a new graph G_1 with vertex-set $V(G[D_1])$ and edge-set $D_1 \cup D_2 \cup D_3$. We know that both $D_1 \cup D_2$ and $D_1 \cup D_3$ induce hamiltonian circuits of G_1 . For $\alpha_{1,1} \in D_2$ and $\gamma_1 \in D_3$, we have by Lemma 2.1 that G_1 has two perfect matchings F_1 and F_2 such that $G_1[F_1 \cup F_2]$ has a circuit C'_1 containing $\{\alpha_{1,1}, \gamma_1\}$ and either $F_1 \cap F_2 \subseteq D_3 \subseteq F_1 \cup F_2$ or $F_1 \cap F_2 \subseteq D_1 \subseteq F_1 \cup F_2$.

Set $E_2 := (D_1 \cap (F_1 \triangle F_2)) \cup (\bigcup_{j \in [q] \text{ s.t. } \alpha_{1,j} \in F_1 \triangle F_2} E(P_{1,j})) \cup (\bigcup_{j \in [q] \text{ s.t. } \gamma_j \in F_1 \triangle F_2} E(Q_j))$. From the properties (2), (3) and (4) of C , we can see that Q_1 has even length and Q_i has odd length for each $i \in [q] \setminus \{1\}$. Hence we can obtain that every component of $G[E_2]$ is an even circuit of G and E_2 can be decomposed into two matchings N_4 and N_5 of G .

Assume $F_1 \cap F_2 \subseteq D_3 \subseteq F_1 \cup F_2$. Set $N_6 := (\bigcup_{j \in [q] \text{ s.t. } \alpha_{1,j} \in E(G_1) \setminus (F_1 \cup F_2)} (E(P_{1,j}) \cap M_1)) \cup (\bigcup_{j \in [q] \text{ s.t. } \gamma_j \in F_1 \cap F_2} (E(Q_j) \setminus M_2))$. We can obtain $(M_1 \cap (E(C) \cup E(C_2))) \setminus (E_2 \cup N_6) \subseteq D_1$ and that N_6 is a perfect matching of $G[E(C) \cup E(C_2)] - V(G[E_2])$.

For $i = 4, 5$, set $M_i := N_{i-2} \cup N_i \cup N_6$. Then M_4 and M_5 are perfect matchings of G and we have $M_1 \setminus (M_4 \cup M_5) \subseteq (E(C_1) \setminus (E(C) \cup N_2 \cup N_3)) \cup D_1$. From the properties (2), (4) and (5) of C and C' , we can obtain that $(M_1 \triangle (E(C') \cup E(C_4))) \cup \{u_1 u_2\}$ is a perfect matching containing $(E(C_1) \setminus (E(C) \cup N_2 \cup N_3)) \cup D_1$ in G . So the edges M_1 can be covered by 3 perfect matchings of G , which together with M_2 and M_3 cover all edges of G .

Assume $F_1 \cap F_2 \subseteq D_1 \subseteq F_1 \cup F_2$. Set $N_7 := (\bigcup_{j \in [q] \text{ s.t. } \alpha_{1,j} \in E(G_1) \setminus (F_1 \cup F_2)} (E(P_{1,j}) \cap M_1)) \cup (F_1 \cap F_2) \cup (\bigcup_{j \in [q] \text{ s.t. } \gamma_j \in E(G_1) \setminus (F_1 \cup F_2)} (E(Q_j) \cap M_2))$. We have that N_7 is a perfect matching of $G[E(C) \cup E(C_2)] - V(G[E_2])$.

For $i = 6, 7$, set $M_i := N_{i-4} \cup N_{i-2} \cup N_7$. Then M_6 and M_7 are perfect matchings of G . We can see $(M_2 \cap E(C_1)) \cup (M_1 \cap E(C_2)) \subseteq M_6 \cup M_7$. Noting $E(Q) \subseteq E(Q_1)$ and $\gamma_1 \in E(C'_1)$, we have $E(Q) \subseteq E_2 \subseteq M_6 \cup M_7$. Now we have $E(G) \setminus (M_3 \cup M_6 \cup M_7) \subseteq ((M_1 \cap E(C_1)) \setminus E(Q)) \cup (M_2 \cap E(C_2))$.

Set $M_8 := ((M_1 \cap E(C_1)) \triangle E(C_3)) \cup (M_2 \setminus E(C_1))$. We know $(E(C_5) \cap E(C_1)) \cup (M_2 \cap E(C_2)) \subseteq M_8$. By Claim 1, G has a perfect matching M_9 containing $(M_1 \cap E(C_1)) \setminus (E(Q) \cup E(C_5))$. Now we have $(M_1 \cap E(C_1)) \setminus E(Q) \subseteq M_8 \cup M_9$. So M_3, M_6, M_7, M_8 and M_9 are 5 perfect matchings of G which cover all edges of G . Claim 2 is proved.

Let C_6 be the circuit containing u_1 in $G[E(C_1) \triangle E(C_5)]$. Assume that every circuit different from C_6 in $G[E(C_1) \triangle E(C_5)]$ contains no edges in C_2 . Then $E(C_6) \cap E(C_2) \neq \emptyset$. Let $Q'_1, Q'_2, \dots, Q'_{q'}$ be the paths separated by $E(C_6) \cap E(C_2)$ in C_6 such that $u_1 \in V(Q'_1)$. We can easily check that C_6 and C_3 meet the requirements (1)-(5) in Claim 2. So we know by Claim 2 that G has 5 perfect matchings which cover all edges of G . Next we assume that there is a circuit C_7 different from C_6 in $G[E(C_1) \triangle E(C_5)]$ such that $E(C_7) \cap E(C_2) \neq \emptyset$.

Let $P_{2,1}, P_{2,2}, \dots, P_{2,p}$ be the components in $G[E(C_7) \cap E(C_1)]$. We know that for each $i \in [p]$, $P_{2,i}$ is a M_2 - M_1 alternating path satisfying $|E(P_{2,i}) \cap M_2| = |E(P_{2,i}) \cap M_1| + 1$. For $i = 3, 4$, let $P_{i,1}, P_{i,2}, \dots, P_{i,p}$ be the paths in C_{25-6i} which are separated by $P_{2,1}, P_{2,2}, \dots, P_{2,p}$. We may assume $u_1 \in V(P_{4,1})$. Set $B_i := \{\alpha_{i+1,j} : j \in [p]\}$ for $i = 1, 2, 3$. Now we construct a new graph G_2 whose vertex-set consists of the ends of edges in B_1 and edge-set is $B_1 \cup B_2 \cup B_3$. We know that both $B_1 \cup B_2$ and $B_1 \cup B_3$ induce hamiltonian circuits of G_2 . Set $B'_2 := \{\alpha_{3,j} \in B_2 : E(P_{3,j}) \cap E(C_2) \neq \emptyset\}$.

For $B'_2 \in B_2$ and $\alpha_{4,1} \in B_3$, we have by Lemma 2.1 that G_2 has two perfect matchings F_3 and F_4 such that (1) either $F_3 \cap F_4 \subseteq B_3 \subseteq F_3 \cup F_4$ or $F_3 \cap F_4 \subseteq B_1 \subseteq F_3 \cup F_4$, (2) $G_2[F_3 \cup F_4]$ has a circuit C'_2 containing $\alpha_{4,1}$ such that $B'_2 \cap E(C'_2) \neq \emptyset$ and every circuit different from C'_2 in $G_2[F_3 \cup F_4]$ contains no edges in B'_2 , and (3) if $B_1 \subseteq F_3 \cup F_4$, then G_2 has a circuit C'_3 containing $\alpha_{4,1}$ such that $B_2 \cap E(C'_2) \cap E(C'_3) = \emptyset$, $B_3 \setminus (F_3 \cup F_4) \subseteq B_3 \setminus E(C'_3)$ and $B_3 \setminus E(C'_3)$ is a perfect matching of $G_2 - V(C'_3)$.

Let C_8 be the circuit of G which is obtained from C'_2 by replacing each edge $\alpha_{i,j}$ in C'_2 by the corresponding path $P_{i,j}$. We know $u_1 \in V(C_8)$ and $\emptyset \neq E(C_8) \cap E(C_2) \subseteq E(C_5) \cap E(C_2) \subseteq M_1$. Noting $E(C_5) \cap E(C_4) = \emptyset$, we have $E(C_8) \cap E(C_2) \cap E(C_4) = \emptyset$.

Assume $F_3 \cap F_4 \subseteq B_3 \subseteq F_3 \cup F_4$. We can know that $B_3 \cap E(C'_2)$ is a perfect matching of C'_2 . It implies that $(M_1 \cap E(C_1)) \setminus E(C_8)$ is a perfect matching of $C_1 - (V(C_8) \cap V(C_1))$ and the paths $Q''_1, Q''_2, \dots, Q''_{s'}$ separated by $E(C_8) \cap E(C_2)$ in C_8 satisfy that for each $i \in [s']$, $M_1 \cap E(Q''_i)$ is a perfect matching of $Q''_i - (V(Q''_i) \cap V(C_2))$ if $u_1 \notin V(Q''_i)$. It means that for the perfect matching M_2 of G , C_8 meets the requirements (1)-(3) in Lemma 3.1. By Lemma 3.1, the edges M_1 can be covered by 3 perfect matchings of G , which together with M_2 and M_3 cover all edges of G .

Assume next $F_3 \cap F_4 \subseteq B_1 \subseteq F_3 \cup F_4$. Set $E_3 := \bigcup_{i=2}^4 (\bigcup_{j \in [p] \text{ s.t. } \alpha_{i,j} \in F_3 \triangle F_4} E(P_{i,j}))$. We can know that $(E(C_1) \setminus E_3) \cap M_2$ is a perfect matching of $C_1 - V(G[E_3])$ and every component of $G[E_3 \setminus E(C_8)]$ is an even circuit of G . Noting also that every circuit different from C'_2 in $G_2[F_3 \cup F_4]$ contains no edges in B'_2 , we have that $G[V(C_1)] - (V(C_8) \cap V(C_1))$ has two perfect matchings N_8 and N_9 such that $E(C_1) \setminus (E(C_8) \cup N_8 \cup N_9) = (E(C_1) \setminus E_3) \cap M_1$.

Let $P_{5,1}, P_{5,2}, \dots, P_{5,r}$ be the paths separated by $E(C_8) \cap E(C_2)$ in C_8 such that $u_1 \in V(P_{5,1})$. We can see $E(Q) \subseteq E(P_{5,1})$. Since $F_3 \cap F_4 \subseteq B_1 \subseteq F_3 \cup F_4$, $B_1 \cap E(C'_2)$ is a perfect matching of C'_2 . It implies that for each $i \in [r] \setminus \{1\}$, $M_2 \cap E(P_{5,i})$ is a perfect matching of $P_{5,i} - (V(P_{5,i}) \cap V(C_2))$.

Let C_9 be the circuit of G which is obtained from C'_3 by replacing each edge $\alpha_{i,j}$ in C'_3 by the corresponding path $P_{i,j}$. We know $u_1 \in V(C_9)$ and $E(C_9) \cap E(C_2) \subseteq E(C_5) \cap E(C_2)$. Noting $B_2 \cap E(C'_2) \cap E(C'_3) = \emptyset$, we can obtain $E(C_8) \cap E(C_9) \cap E(C_2) = \emptyset$. Noting $B_3 \setminus (F_3 \cup F_4) \subseteq B_3 \setminus E(C'_3)$ and that $B_3 \setminus E(C'_3)$ is a perfect matching of $G_2 - V(C'_3)$, we can obtain $(E(C_1) \setminus E_3) \cap M_1 \subseteq (M_1 \cap E(C_1)) \setminus E(C_9)$ and that $(M_1 \cap E(C_1)) \setminus E(C_9)$ is a perfect matching of $C_1 - (V(C_9) \cap V(C_1))$. We also can know that $B_3 \cap E(C'_3)$ is a perfect matching of C'_3 . This implies that $E(C_9) \setminus M_1$ is a perfect matching of $C_9 - u_1$.

Now we know that C_8 and C_9 are two circuits of G meeting the requirements (1)-(5) in Claim 2. By Claim 2, G has 5 perfect matchings which cover all edges of G .

□

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